Jamming transition in a homogeneous one-dimensional system: The bus route model

O. J. O'Loan, M. R. Evans, and M. E. Cates
Department of Physics and Astronomy, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom
(Received 19 December 1997)

We present a driven diffusive model that we call the bus route model. The model is defined on a one-dimensional lattice, with each lattice site having two binary variables, one of which is conserved (''buses'') and one of which is nonconserved (''passengers''). The buses are driven in a preferred direction and are slowed down by the presence of passengers who arrive with rate λ. We study the model by simulation, heuristic argument, and a mean-field theory. All these approaches provide strong evidence of a transition between an inhomogeneous ''jammed'' phase (where the buses bunch together) and a homogeneous phase as the bus density is increased. However, we argue that a strict phase transition is present only in the limit $\lambda \to 0$. For small λ, we argue that the transition is replaced by an abrupt crossover that is exponentially sharp in $1/\lambda$. We also study the coarsening of gaps between buses in the jammed regime. An alternative interpretation of the model is given in which the spaces between buses and the buses themselves are interchanged. This describes a system of particles whose mobility decreases the longer they have been stationary and could provide a model for, say, the flow of a gelling or sticky material along a pipe. [S1063-651X(98)01808-X]

PACS number(s): 05.70.Ln, 64.60.−i, 89.40.+k

I. INTRODUCTION AND MODEL

Driven diffusive systems [1] have recently attracted much attention in the field of nonequilibrium statistical mechanics from a fundamental viewpoint, as well as in the context of traffic modeling [2], interface growth [3], and other applications [4].

One particularly interesting feature is the possibility of phase ordering and phase transitions in one-dimensional (1D) systems. To appreciate the significance, one should recall that in 1D equilibrium models, ordering only occurs either in the limit of zero temperature (e.g., kinetic Ising models or deterministic Ginsburg-Landau equation) or in mean-field-like models. However, in nonequilibrium systems it has been demonstrated that ordering may occur in models with fully stochastic, local dynamics [5–7].

The nonequilibrium transitions found so far appear to be of three main types (although very recently, novel phase separation phenomena have been demonstrated in some 1D systems [8,9]). First there are boundary induced transitions [10,6]. These occur on open systems with a dynamics that conserves some quantity in the bulk, but that allows injection and extraction of the quantity at the boundaries. A second class of transition, describing roughening of a 1D interface, is connected to directed percolation and corresponds to a driven diffusive system with nonconserved order parameter [11,12]. Finally, there are transitions induced by defect sites [13,14] or particles [15–18] or the presence of disorder [19–21]. In this class of systems, the presence of the defect causes a macroscopic region of high density to form. Analogies with Bose-Einstein condensation [19] and liquid-gas phase coexistence [17] have been made. An even simpler way to view the phenomenon is as a jamming transition; the defect causes a traffic jam to form behind it. In this context, however, ''jamming'' may be a somewhat misleading term, since in the models just described the inhomogeneous ''jammed'' phase arises at *low* density. The transition is in fact between this phase, and a higher density ''congested'' phase, which is uniform, but which has a mean particle velocity *lower* than that of the jammed phase. Indeed, it appears that a minimum velocity principle applies [19], so that the stable phase is always the slowest available at a given mean density.

In this work we address the question of whether similar ''jamming'' transitions can occur in 1D homogeneous systems, i.e., systems with periodic boundary conditions and without disorder or defects. We introduce a model that exhibits a jamming transition in a certain limit (to be specified below). For the moment, it is useful to describe the model in terms of a commonly experienced and universally irritating situation. Consider buses moving between bus stops along a bus route. Clearly, the ideal situation is that the buses are evenly distributed along the route so that each bus picks up roughly the same number of passengers. However, owing to some fluctuation, it may happen that a bus is delayed and the gap to the bus in front of it becomes large. Then, the time elapsed since the bus stops in front of the delayed bus have been visited by the previous bus is larger than usual and consequently more passengers will be waiting at these bus stops. Therefore the bus becomes delayed even further. At the same time, the buses behind catch up with the delayed bus and pick up only very few passengers since the delayed bus takes them all. Hence a ''jam'' of buses forms. Inspired by this scenario we shall formulate a model below, to be referred to as the bus route model (BRM).

We defer the mathematical definition of the BRM until after we have discussed the general context. Already, from the simple picture discussed in the previous paragraph, we can identify a conserved variable (the buses) and a nonconserved variable (the passengers). The passengers are nonconserved since they arrive at the bus stops from outside the system (bus route). The conserved buses are driven in a preferred direction. We may usefully think of the nonconserved variable coupling to the conserved variable and mediating the jamming transition.

Associated with any ordering dynamics is the phenomenon of coarsening [22] where the typical domain length of the ordered phase grows indefinitely with time. Indeed, coarsening has been studied in ballistic aggregation models [23] and disordered driven diffusive systems where jamming occurs [20,24]. Again, a contrast can be made with 1D equilibrium models where only zero temperature models or mean-field-like models coarsen. In the present model, it is the gaps between the jams that coarsen as the jams aggregate; we study this phenomenon in the present work.

In a finite system, the coarsening eventually results in one large jam with a single gap in front of it. Recalling that the model system is homogeneous and that no bus is preferred over any other, we see that we have a spontaneous symmetry breaking where the symmetry between buses is broken through one bus being selected to head the jam. Symmetry breaking transitions have been previously found in 1D open systems with a nonconserved variable at the boundaries [6,7] and in a class of growth models [11]. The present model provides an example in a homogeneous system with a conserved variable. Related symmetry breaking has also been noted in some models in a class of ''backgammon'' or ''balls in boxes'' models that are effectively simple generalizations of Bose systems [25–27]. However, in these models the dynamics is inherently equilibrium and mean-field-like whereas the BRM has local dynamics that does not satisfy detailed balance. We shall elucidate the connection between the two classes of models by showing that a mean-field approximation to the BRM results in a model that may be solved analytically. The steady state of this soluble mean-field model falls into the class of generalized Bose systems.

We now formally define the BRM [38]. The model is defined on a 1D lattice with periodic boundary conditions. Each lattice site is labeled by a number i running from 1 to L. Site i has two binary variables τ_i and ϕ_i associated with it. These variables can be described in the following terms: (i) If site i is occupied by a bus then $\tau_i=1$; otherwise $\tau_i=0$. (ii) If site i has passengers on it then $\phi_i=1$; otherwise $\phi_i=0$. Each site can be thought of as a bus stop on a bus route. A site cannot have both $\tau_i=\phi_i=1$ (i.e., it cannot have a bus *and* passengers).

There are M buses and L sites in the system and the bus density

$$\rho=M/L \tag{1}$$

is a conserved quantity. However, the total number of sites with passengers is *not* conserved. The update rules for the system are as follows: (1) Pick a site i at random. (2) If $\tau_i=0$ and $\phi_i=0$ then $\phi_i\to 1$ with probability λ. (3) If $\tau_i=1$ and $\tau_{i+1}=0$, define a hopping rate μ as follows: (i) $\mu=\alpha$ if $\phi_{i+1}=0$; (ii) $\mu=\beta$ if $\phi_{i+1}=1$, and update $\tau_i\to 0$, $\tau_{i+1}\to 1$ and $\phi_{i+1}\to 0$ with probability μ.

Thus, α is the hopping rate of a bus onto a site with no passengers and β is the hopping rate onto a site with passengers. The probability that a passenger arrives at an empty site is λ. When a bus hops onto a site with passengers, it removes the passengers. While we have taken the passenger variable ϕ_i to be binary, this does not forbid the presence of more than one passenger at a site; we merely require that the extra passengers have no further effect on the dynamics. We generally take $\beta<\alpha$, reflecting the fact that buses are slowed down by having to pick up passengers. We may set α equal to 1 without loss of generality and from now on we consider only this case. We note that the dynamics is local and does not satisfy detailed balance. The model is illustrated schematically in Fig. 1.

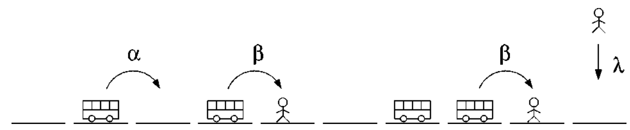

FIG. 1. Schematic illustration of the BRM.

Although the language of ''buses'' and ''passengers'' provides an appealing mental picture, the model is intended to be simple rather than realistic. For example, the need for buses to stop to allow passengers to disembark is ignored. Note, however, that an ability for buses to overtake each other would have almost no effect. This is because, in a jammed situation of the type discussed below, the interchange of a fast-moving bus with a slower moving one in front also interchanges their velocities.

The main purpose of this paper is to provide evidence that the model defined above undergoes a jamming transition. However, we argue that a strict transition (i.e., a singularity in some measured quantity) only occurs as $\lambda\to 0$ and in the thermodynamic limit. Here we define the thermodynamic limit as

$$M,L\to\infty \quad \text{with } \rho \text{ held fixed.} \tag{2}$$

Our evidence is both numerical and analytical. In Sec. II we present Monte Carlo simulations that provide evidence for the above picture. In the following sections we provide various analyses to support the picture. In particular, in Sec. II A we present a simple two-particle approximation that describes the stability of jams. In Sec. III we define a mean-field approximation. Within this approximation we can solve analytically for the steady state and find it is similar to a generalized ideal Bose gas or backgammon model. This steady state can be analyzed and exhibits a phase transition (taking the form of a condensation transition) only as $\lambda\to 0$ in the thermodynamic limit. We show that the mean-field steady state agrees quantitatively with Monte Carlo simulations of the BRM, suggesting that for the BRM also there is no strict transition for nonzero λ. We also study numerically the behavior as $\lambda\to 0$ in the mean-field approximation.

In Sec. IV we study the approach to the steady state in the jammed phase where we observe coarsening of bus clusters. We argue that on an infinite system the size of the large gaps between clusters should eventually grow as $t^{1/2}$. We study finite systems numerically. In Sec. V we discuss an interpretation of the BRM wherein we consider the vacant sites *between* buses to be the moving entities in the system. Viewed this way, the nonconserved variable now describes an internal degree of freedom of the moving entities themselves, rather than of the sites they visit. We discuss possible physical interpretations of this dynamics, including a model of clogging. In Sec. VI, we consider the relevance of the BRM to real bus routes and in Sec. VII, we conclude with a discussion of the main points of our work.

II. SIMULATION RESULTS AND HEURISTIC ARGUMENTS

The model defined above captures an important feature of the bus route problem described in the Introduction, namely, that once a gap between buses becomes large through some fluctuation, the tendency is for the gap to become still larger: since buses move more quickly in areas with few passengers, a bus that is following closely behind another will tend to move faster than one that is a long way behind the bus directly in front. This is simply because the closer a bus is to the one directly ahead of it, the less time passengers will have had to arrive. If this tendency for large gaps to grow were to prevail, the result on a finite system would be a single jam of buses and one large gap. We first argue that in the limit of $\lambda \to 0$ but $\lambda L \to \infty$ this scenario can only hold at low enough density of buses, and that a phase transition to a homogeneous phase will result as the density is raised.

First consider a system comprising a single large jam. In order for this to be a stable object, the velocity (defined as the average rate of hopping forward) of the leading bus must be equal to the velocity of any bus inside the jam. Now if $\lambda L \to \infty$, the probability that the site immediately in front of the leading bus has a passenger on it tends to one. (This is because the rate of passenger arrival, multiplied by the time delay between the final bus of the jam and the leading bus of the jam crossing the same site, is of order λL.) Therefore the velocity of the leading bus will approach β. On the other hand, if $\lambda \to 0$ the probability that a site within the jam has passengers tends to zero since these gaps are of finite length. Therefore, since the velocity of a bus in the jam will only be limited by the presence of neighboring buses, this velocity will be $1-\rho_{\mathrm{jam}}$ where ρ_{jam} is the density of buses *within* the jam. (This follows from the fact that, when $\lambda \to 0$ with $\lambda L \to \infty$, the situation is equivalent to a model of hopping particles with a single slow ''defect'' particle [19].) Equating the two velocities yields $\rho_{\mathrm{jam}} = 1-\beta$. However, the jam must clearly have density greater than ρ, the overall density of the system. Thus for jamming to occur we require

$$\rho < \rho_c \quad \text{where} \quad \rho_c = \rho_{\mathrm{jam}} = 1-\beta. \tag{3}$$

For densities above this critical value, the system will be in a congested phase where gaps between buses are uniform.

An equivalent way of obtaining Eq. (3) is to compare the velocity in the jammed phase, β, and the velocity in the homogeneous phase, $1-\rho$. The phase with the lowest velocity is chosen by the system. This procedure is analogous to the thermodynamic procedure of choosing the phase with the lowest chemical potential. Though unproven for the present problem, in the case of disordered exclusion processes the analogy has been shown to be exact [19].

We now present simulations that, for small λ, qualitatively support the above picture of a phase transition. Figure 2 shows a space-time plot of the buses in the system at $\rho = 0.55$ for small λ ($\lambda = 0.02$). The buses are distributed fairly homogeneously throughout space. There are no very large gaps present in the system. Figure 3 shows a space-time plot for $\rho = 0.2$, which is less than ρ_c of Eq. (3). In this case, starting from a random configuration of buses, large gaps quickly open up and small clusters or ''jams'' of buses are readily seen. Gradually, these small jams coarsen until, finally, the system comprises a single large jam. There is one large interbus gap in front of the jam whereas the jam itself contains many small gaps. The behavior can be thought of as phase separation into regions of nonzero and zero density.

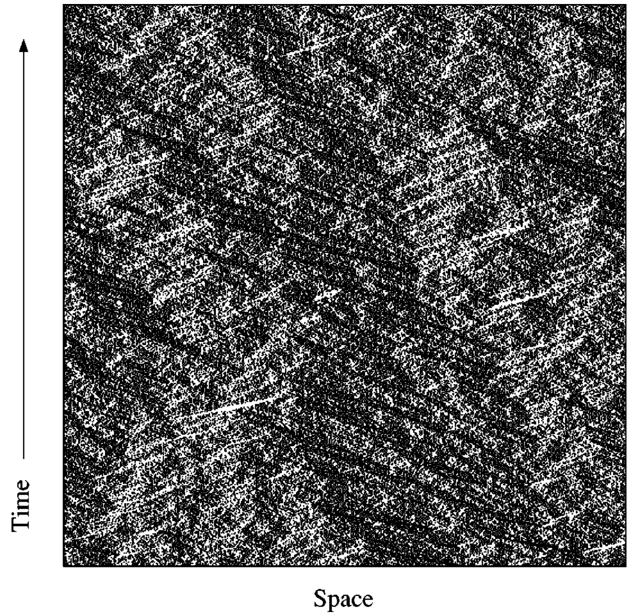

FIG. 2. Space-time plot of bus positions for $\lambda = 0.02$, $\rho = 0.55$, $\beta = 0.5$, and $L = 500$. There are 10 time steps between each snapshot on the time axis. Initially the buses are positioned randomly and there are no passengers.

In order to investigate the effect of varying λ, we plot in Fig. 4 the steady state average velocity v as a function of the density. A system size $L = 10\,000$ was chosen since using a bigger system did not appreciably affect the results. Let us first consider the results for $\beta = 0.5$. For $\lambda = 0.1$, the velocity increases smoothly as the density is decreased and appears to approach β for small density. There is no sign of a phase

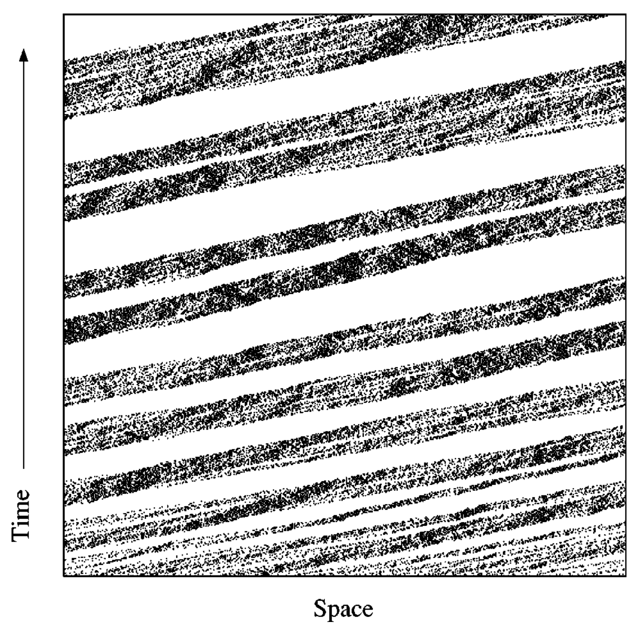

FIG. 3. Space-time plot of bus positions for the same parameters as in Fig. 2 except that here, $\rho = 0.2$.

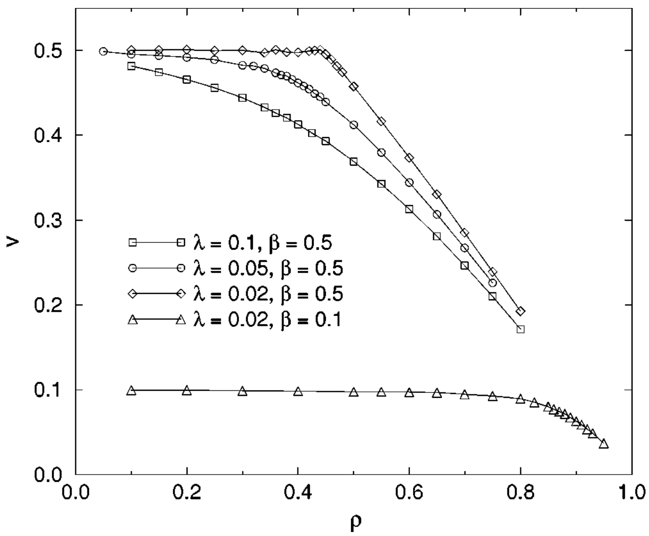

FIG. 4. Velocity as a function of density for various values of λ and β. The simulations were performed with $L=10\,000$. The lines are shown to guide the eye.

transition. For $\lambda=0.05$ a similar picture holds although now the curve $v(\rho)$ is more concave.

A strikingly different picture is obtained for $\lambda=0.02$. There is an apparent discontinuity in the derivative of $v(\rho)$ at some value of the density $\rho^*\simeq 0.45$. Below ρ^*, the data are consistent with $v=\beta$ whereas above ρ^*, v decreases almost linearly with increasing density. This behavior is consistent with the simple picture of a jamming transition discussed above. It is also very similar to the velocity-density relationship in an exactly solvable model with a single slow particle [19,20] where a jamming transition occurs. The graph therefore suggests that a phase transition may occur for small λ. We argue, however, that the phase transition occurs only in the limit $\lambda\to 0$ and that this limit has to be taken in an appropriate way. In order to quantify this we now analyze a simple two-particle approximation to the full system.

A. Two-particle approximation

Consider a system containing M buses. Let us assume that there is a jam in the system [i.e., a gap with size $O(L)$]. If there is a jamming transition then such a gap should become stable in the thermodynamic limit; the bus at the head of the jam should not be able to escape into the large gap. Let us consider the two buses A and B at the head of such a jam as shown in Fig. 5. The gap in front of A has size $kL-x$ where k is independent of L. The gap in front of B has size x. We now assume that we can write the hop rate of either bus as a function only of the gap size in front of that bus. To do this we write $u(x)=f(x)+\beta[1-f(x)]$ where $f(x)$ is the probability that there are no passengers on the first site of a gap of size x. To estimate $f(x)$ we assume that bus A is a random

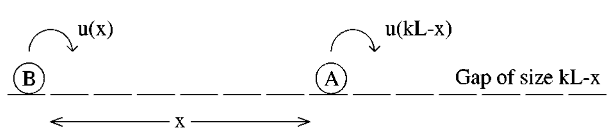

FIG. 5. The two buses at the head of a jam in a system of size L.

walker hopping with rate v, where v is the average velocity of the system. Then the average time since A left the site in front of B is x/v. Since the rate of arrival of passengers is λ we then have $f(x)=\exp(-\lambda x/v)$ and

$$u(x)=\beta+(1-\beta)\exp(-\lambda x/v) \quad \text{for } x>0, \qquad u(0)=0. \tag{4}$$

The use of expression (4) for the hopping rates is in the spirit of a mean-field approximation. We have found by simulation that the approximation is a good one for β larger than about 0.2, but for small β it breaks down. In particular, for small λ (when β is small) we find that $u(x)$ decays to β much more rapidly than Eq. (4) predicts. We believe that the reason for this is the failure of Eq. (4) to take into account time correlations in the hopping of buses—when a bus is updated and fails to hop into an unoccupied site, the next time it is updated it should hop with probability β and *not* $u(x)$ (because for a bus to fail to hop, the site ahead *must* contain passengers; this is no longer true for $\alpha\neq 1$, but time correlations will still be present). Clearly, the effects of neglecting these time correlations will be largest for small β. We shall return to Eq. (4) in Sec. III where we carry out a conventional mean field theory for the many-particle problem. For our present purposes, we replace v by β in Eq. (4) since we are interested in a jammed situation (i.e., we assume that bus A always has passengers to pick up).

Using the mean-field hopping rate $u(x)$ we may write a Langevin equation for the dynamics of the gap size:

$$\dot{x}=u(kL-x)-u(x)+\eta(t), \tag{5}$$

where $\eta(t)$ is a noise term (say white noise of unit variance). [The variance of the noise should strictly depend on β but, since we are primarily interested in the effect of (small) λ on the dynamics of the gap, we ignore this dependence.] This can be written in the form

$$\dot{x}=-\frac{d}{dx}\Phi(x)+\eta(t), \tag{6}$$

where

$$\Phi(x)=-\frac{\beta(1-\beta)}{\lambda}[e^{-\lambda(kL-x)/\beta}+e^{-\lambda x/\beta}]. \tag{7}$$

The gap size x has the dynamics of a particle undergoing diffusion in a potential $\Phi(x)$ for $x>0$. There is a reflecting boundary at $x=0$ and the potential has a maximum at $x^*=kL/2$. Therefore for $x<x^*$ the particle is trapped in a well $(0<x<x^*)$ and bound to the reflecting boundary, i.e., A is bound to the head of the jam. If $x>x^*$, the particle has escaped from the well and A is no longer at the head of the jam. The well depth $\Delta\Phi=\Phi(x^*)-\Phi(0)$ is given by

$$\Delta\Phi=\frac{\beta(1-\beta)}{\lambda}[1-e^{-\lambda kL/2\beta}]^2. \tag{8}$$

When the thermodynamic limit $L\to\infty$ is taken, $\Delta\Phi\to\beta(1-\beta)/\lambda$, which is finite for $\lambda>0$, indicating that the jam is unstable. However, when the limit $\lambda\to 0$ is now taken, $\Delta\Phi$

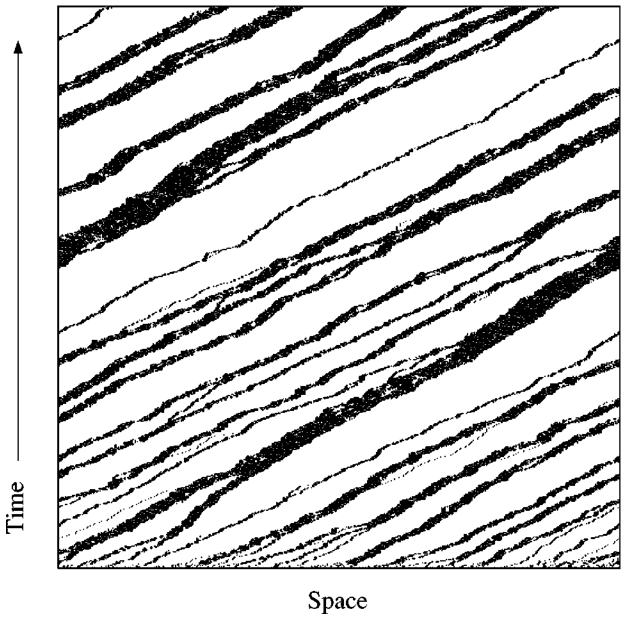

FIG. 6. Space-time plot of bus positions for $\lambda=0.02$, $\rho=0.2$, $\beta=0.1$, and $L=500$. There are 20 time steps between each snapshot on the time axis. Initially the buses are positioned randomly and there are no passengers.

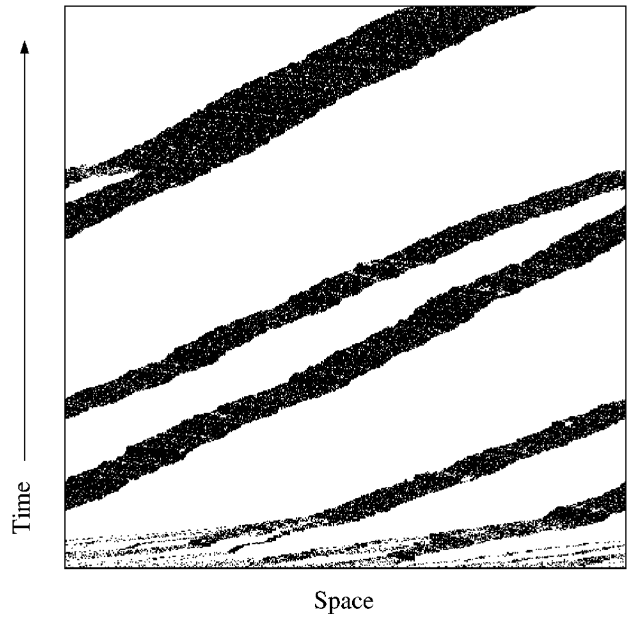

FIG. 7. Space-time plot of bus positions for the same parameters as in Fig. 6 except that here, $\lambda=0.002$.

diverges and particle A becomes bound to particle B—in this limit the jam is a stable object. This simple analysis suggests that there is a phase transition (between a jammed and a homogeneous phase as the density is raised) only in the limit $\lambda\rightarrow0$. For small λ, the time scale associated with the detachment of particle A from the head of the jam is, to a first approximation [28], given for large L by

$$\tau\sim L^2\exp(\Delta\Phi)=L^2\exp\left[\frac{\beta(1-\beta)}{\lambda}\right]. \tag{9}$$

The prefactor L^2 reflects the facts that the well width depends linearly on L and that, for nonzero λ, particle A escapes diffusively. When λ is small but nonzero, τ is exponentially large in $1/\lambda$ and it can appear that a jam is stable when in fact it is not.

It is clear that the limits $\lambda\rightarrow0$ and $L\rightarrow\infty$ do not commute since as $\lambda\rightarrow0$ on a finite system one recovers a model where all hopping rates are the same. Such a model of hopping particles with hard core exclusion (known as an asymmetric exclusion process) has been well studied and exhibits no phase transition with periodic boundary conditions [1]. In practice, one could take the limit $L\rightarrow\infty$ and then $\lambda\rightarrow0$ by choosing

$$\lambda\sim L^{-\gamma}\quad\text{with}\quad\gamma<1 \tag{10}$$

and taking the thermodynamic limit $L\rightarrow\infty$ whereupon the escape time diverges exponentially in the system size:

$$\tau\sim L^2\exp(aL^{\gamma}) \tag{11}$$

and the jam is a stable object.

In order to test this picture against simulations, we plot in Fig. 6 a space-time plot for $\beta=0.1$ and $\lambda=0.02$ for a low density. At first glance, it appears that jamming just like that seen for $\beta=0.5$ is taking place. However, closer inspection of the individual jams shows that they can develop large gaps and, in some cases, divide into smaller jams. The steady state for this system is therefore not characterized by a single large jam (as appeared to be the case from simulations for $\beta=0.5$), but comprises a number of jams of varying sizes.

Figure 7 shows a space-time plot for the same value of β (0.1) but for $\lambda=0.002$, a factor of 10 smaller. It appears that the jamming has been restored—the individual jams are not seen to break up. It is also interesting to note that splitting of jams is not observed for $\beta=0.5$ (Fig. 3) so the splitting in Fig. 6 is a direct consequence of having a smaller value of β. Figure 4 also shows $v(\rho)$ for $\lambda=0.02$ and $\beta=0.1$. Unlike the case $\beta=0.5$, $v(\rho)$ is a smooth function with apparently no discontinuity in its derivative. This is consistent with the observation that the jammed state is unstable on the time scale of the simulation (see Fig. 6).

We therefore conclude from our two-particle argument that although the simulation data are indicative of the presence of a phase transition, it cannot be taken as firm evidence that a strict transition occurs for any nonzero λ. However, for small λ something closely approaching a transition is seen.

III. MEAN-FIELD MODEL

In this section we discuss a mean-field theory that we believe describes the behavior of the BRM very well. Motivated by the mean-field probability (4) of hopping into a gap of a given size, derived in Sec. II A, we approximate the BRM by a model that is exactly solvable in the steady state. We term this new model the *mean-field model* (MFM). We analyze this MFM and show that it compares quantitatively well with the BRM and, importantly, that it does not have a phase transition for $\lambda>0$. However, we find that there is a transition in the MFM in the limit $\lambda\rightarrow0$ so long as the thermodynamic limit is taken first. This supports our view that the same applies in the BRM itself.

We define the MFM to be a hopping particle model (generalized asymmetric exclusion process [1]) with the hopping probability of the ith particle being $u(x_i)$, a function of x_i, the size of the gap in front of that particle. As in the BRM, we consider M particles and L sites in total, and periodic boundary conditions apply. The hopping probabilities $u(x)$ are given by the mean-field expression (4). Note that $u(x)$ depends on v, which is the steady state mean velocity of the system, $\langle u\rangle$. It is useful to think of v in Eq. (4) as an adjustable parameter in the MFM to be determined self-consistently; i.e., v is chosen (for fixed ρ, β, and λ) so that $\langle u\rangle = v$.

While $u(x)$ is given by Eq. (4) for the MFM, it is convenient to first consider a more general $u(x)$, with the only constraints being $u(0)=0$ (exclusion) and $u(x)\to\beta^+$ as $x\to\infty$. When one considers the dynamics of the gaps between particles, it is evident that such a model is an example of what is known in the mathematical literature as a zero-range process [29,30], which is exactly soluble (in the steady state). We present the solution in detail in the Appendix and quote the result here. Each configuration can be uniquely identified by a set of gap sizes $\{x_i\}=\{x_1,x_2,\ldots,x_M\}$. The steady-state probability of a configuration $\{x_i\}$ is

$$P(\{x_i\})=\frac{1}{Z(L,M)}q(x_1)q(x_2)\cdots q(x_M), \tag{12}$$

where

$$q(x)=\prod_{y=1}^{x}\frac{1}{u(y)} \tag{13}$$

and

$$Z(L,M)=\sum_{x_1,x_2,\ldots,x_M} q(x_1)\cdots q(x_M)\,\delta_{x_1+\cdots+x_M,L-M}. \tag{14}$$

$Z(L,M)$ can be viewed as the partition function of a generalized noninteracting system of Bose particles. We follow Bialas, Burda, and Johnston [26] in the consequent analysis, starting from the expression for $Z(L,M)$. We first discuss the behavior for general $u(x)$ in the thermodynamic limit and state the conditions that $u(x)$ must satisfy in order for a phase transition to occur. Turning to the specific case of the MFM, we show that there is a phase transition in the limit $\lambda\to 0$. We describe a method for analyzing finite systems and we compare the MFM with simulation results for the BRM. Finally, we discuss the behavior for small, but nonzero, λ.

A. Mean-field model in the thermodynamic limit

Using the integral representation of the Kronecker delta

$$\delta_{m,n}=\oint\frac{ds}{2\pi i}\frac{s^m}{s^{n+1}}$$

we write $Z(L,M)$, defined in Eq. (14), as

$$Z(\rho,M)=\oint\frac{ds}{2\pi i s}\left[\frac{F(s)}{s^{1/\rho-1}}\right]^M, \tag{15}$$

where $\rho=M/L$ and we have defined the generating function

$$F(s)=\sum_{x=0}^{\infty} q(x)s^x. \tag{16}$$

The integral (15) is calculated in the thermodynamic limit [defined in Eq. (2)] using the saddle point method. It can be shown that the saddle point, where $s=z$, is given through the expression

$$\frac{1}{\rho}-1=zg(z), \tag{17}$$

where

$$g(z)=\frac{F'(z)}{F(z)}. \tag{18}$$

We may identify z as the fugacity. Each value of the fugacity gives a particular value of the density. The partition function becomes

$$Z(\rho,M)\sim\exp\{M\ \ln F(z)-(L-M)\ln(z)\} \tag{19}$$

with z for a particular ρ determined by solving Eq. (17).

To obtain quantitative results from this solution, it is useful to obtain an expression for $p(x)$, the steady state probability that a given particle has a gap of size x in front of it. For any L and M, $p(x)$ is given by

$$p(x)=\frac{q(x)}{Z(L,M)}\sum_{x_2,\ldots,x_M} q(x_2)\cdots q(x_M)\,\delta_{x_2+\cdots+x_M,L-M-x}$$
$$=q(x)\frac{Z(L-x-1,M-1)}{Z(L,M)}. \tag{20}$$

This can be determined for a finite gap size x in the thermodynamic limit using the saddle-point expression for $Z(L,M)$ given in Eq. (19). We obtain

$$p(x)=\frac{q(x)z^x}{F(z)}. \tag{21}$$

This expression is in terms of z but ρ may be found using Eq. (17).

The mean particle velocity v in the steady state is

$$v=\langle u\rangle=\sum_{x=1}^{\infty}u(x)p(x). \tag{22}$$

Substituting Eq. (21) into Eq. (22) gives the result that $v=z$. This is a relationship that has been found before in this kind of system [19]. We show below that z is constrained to be no greater than β and hence, there is an upper limit on the velocity. This constraint on z combined with Eq. (21) implies that in the thermodynamic limit, $p(x)$ is a monotonically decreasing function of x and behaves for large x as $p(x)\propto(z/\beta)^x$.

We now examine the criteria that $u(x)$ must satisfy for a phase transition to occur. This entails analysis of the generating function $F(z)$ defined by Eq. (16). $F(z)$ converges for $z<\beta$ and diverges for $z>\beta$ [since we have $u(x)\to\beta^+$ as $x\to\infty$]. Now, $g(z=0)=0$, which corresponds to $\rho=1$. Observe that $g(z)$ is a monotonically increasing function for $0\leqslant z\leqslant\beta$, which means that ρ decreases monotonically from 1 to $1/[1+\beta g(\beta)]$ for z in this range. It is clear then that $g(\beta)$, if finite, will give a critical density, ρ_c, via Eq. (17). Indeed, Bialas, Burda, and Johnston [26] show that in this case there is a transition from a high density "congested" phase to an inhomogeneous "jammed" phase for $\rho<\rho_c$ where a single gap will occupy a finite fraction of the sites in the system. The critical density is given by

$$\rho_c=\lim_{z\to\beta^-}\frac{1}{1+zg(z)}. \tag{23}$$

For a transition to occur at nonzero density, we must have $g(z)$ [and hence both $F(z)$ and $F'(z)$] convergent as $z\to\beta^-$.

For convenience, we write $u(x)=\beta[1+\zeta(x)]$. By considering the asymptotic behavior of the product $q(x)$ in the large x limit, one finds that a phase transition occurs (i.e., ρ_c is nonzero) if and only if $\zeta(x)$ decays to zero more slowly than $2/x$ as $x\to\infty$. For the MFM, where $u(x)$ is given by Eq. (4), we have $\zeta(x)\sim\exp(-\lambda x)$ and we immediately see that there is no transition for $\lambda>0$. We now show that, in contrast, in the limit $\lambda\to 0^+$, a transition does occur.

For the MFM, $F(z)$ is given by

$$F(z)=\sum_{x=0}^{\infty}\left(\frac{z}{\beta}\right)^x\prod_{y=1}^{x}\frac{1}{1+\zeta(y)}, \tag{24}$$

where $\zeta(x)=(1/\beta-1)\exp(-\lambda x/v)$. To show that a transition occurs in the limit $\lambda\to 0^+$, we must show that

$$\lim_{z\to\beta^-}\lim_{\lambda\to 0^+}F(z)\quad\text{and}\quad\lim_{z\to\beta^-}\lim_{\lambda\to 0^+}F'(z) \tag{25}$$

both converge and also that

$$\lim_{z\to\beta^+}\lim_{\lambda\to 0^+}F(z)=\lim_{z\to\beta^+}\lim_{\lambda\to 0^+}F'(z)=\infty. \tag{26}$$

Since for $\lambda>0$ both $F(z)$ and $F'(z)$ diverge when $z\geqslant\beta$, the requirement (26) is satisfied trivially. Note that if the limit $\lambda\to 0$ is taken *before* the thermodynamic limit, values of the fugacity greater than β are permitted and we recover a model where all particles hop with probability 1.

We now show that the expressions in Eq. (25) are finite and we calculate the critical density. Clearly, $F(z)$ and $F'(z)$ both converge for $\lambda\geqslant 0$ when $z<\beta$ [since the geometric part $(z/\beta)^x$ dominates the series for large x]. Since $1+\zeta(x)=1/\beta$ for $\lambda=0$, one can easily calculate $F(z)=\Sigma z^x$ and $F'(z)=\Sigma xz^{x-1}$. Using the saddle-point equation (17), one finds that the density ρ is $1-z$. Therefore, the critical density is nonzero and is given by $\rho_c=1-\beta$. We anticipated this result in the discussion in Sec. II in the context of the $\lambda\to 0$ transition in the BRM. Note that the analysis given here does not make any predictions about the behavior for densities below ρ_c. However, the results presented in Secs. III C and III D, together with the discussion in the context of the BRM in Sec. II, lead to the conclusion that the low density phase comprises a jam in which the leading particle hops forward with probability β (because the site in front of it contains passengers but no bus), and all following particles hop forward with probability 1, so long as the site in front contains no bus (which holds with probability β).

B. Finite systems

It is possible to analyze the MFM for finite systems since, using Eq. (20), we have

$$\sum_{x=0}^{L-M}p(x)=1=\sum_{x=0}^{L-M}\frac{Z(L-x-1,M-1)q(x)}{Z(L,M)},$$

which gives the following recurrence relation for $Z(L,M)$:

$$Z(L,M)=\sum_{x=0}^{L-M}Z(L-x-1,M-1)q(x) \tag{27}$$

with the initial condition $Z(L,1)=q(L-1)$.

In principle, one can calculate $Z(L,M)$ for any L and M. However, numerical precision restricts the practical maximum values of L and M to ~ 2000. In practice for these finite system calculations, it is not feasible to retain the dependence of $u(x)$ on v and so we simply take $v=\beta$ in the expression (4) for $u(x)$. This is a good approximation since we are principally interested in the case where v is close to β.

C. Comparison with BRM

We now proceed to compare MFM results with BRM simulation data. All the results presented for the MFM are based on a calculation of the gap size distribution $p(x)$. It is not possible to obtain a closed analytic expression for $p(x)$ and so one must perform the calculation numerically for a given pair of parameters β and λ.

Figure 8 shows $v(\rho)$ for the BRM and the MFM; the agreement is quite good. For $\lambda=0.1$, we have found numerically that in the BRM, system size is not an important factor for large enough systems and $v(\rho)$ for $L=1000$ is essentially the same as $v(\rho)$ in the thermodynamic limit. [The major part of the small discrepancy between the MFM data for $L=1000$ and $L=\infty$ at high density can be attributed to the fact that the calculation for the finite system is performed with v being replaced by β in the expression for $u(x)$.]

Recall from Fig. 4 that in the BRM, where we had $L=10\,000$, for $\lambda=0.02$ and $\beta=0.5$, $v(\rho)$ exhibited an apparent discontinuity in its derivative, a possible signal of a phase transition. For the MFM in the thermodynamic limit with the same parameters, $v(\rho)$ is very similar (see Fig. 8) but we know that it is actually a smooth function and that there is no transition. This is consistent with our view that there is in fact no phase transition in the BRM for $\lambda>0$. Figure 8 also shows that, for this value of λ (unlike $\lambda=0.1$), there is a marked finite size effect at $L=1000$ comprising a bump in $v(\rho)$. The bump appears for both the BRM numerics and the MFM calculation. The reason for the presence of the bump is that the size of the large gap in front of a jam decreases as L

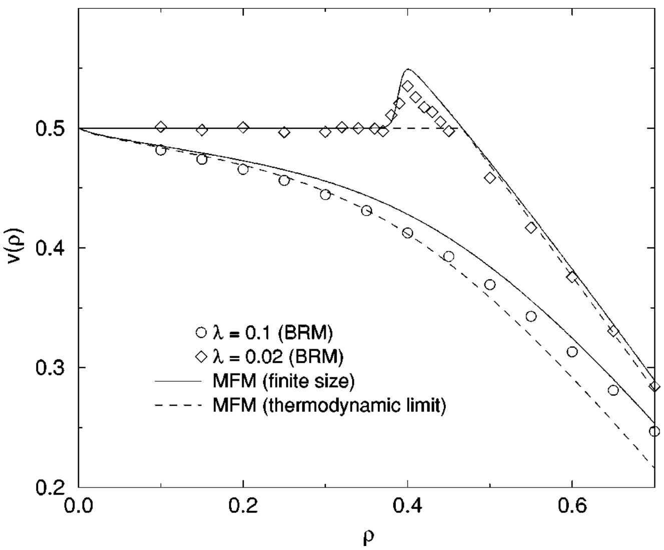

FIG. 8. Velocity as a function of density for the BRM (simulation) and the MFM (calculation) for $\lambda=0.1$ and $\lambda=0.02$. β is 0.5 and L is 1000. MFM data for the thermodynamic limit is also shown.

decreases, resulting in the ''head'' of the jam catching up with the ''tail'' (and hence an average velocity greater than β) for densities sufficiently close to $\rho_c=1-\beta$.

Figure 9 compares the BRM and MFM gap size distributions for small systems in the jammed regime. The distributions are bimodal with the approximately Gaussian second peak corresponding to the presence of a single large gap in the system. The agreement between simulation and MFM is again reasonably good. The MFM distribution in the thermodynamic limit is also shown and one can see that it decays monotonically, but very slowly, for large gap sizes. We have found that the position of the peak in the tail of $p(x)$ increases linearly with system size as one would expect—the peak corresponds to the presence of an ''extensive'' gap (a gap with size $\propto L$).

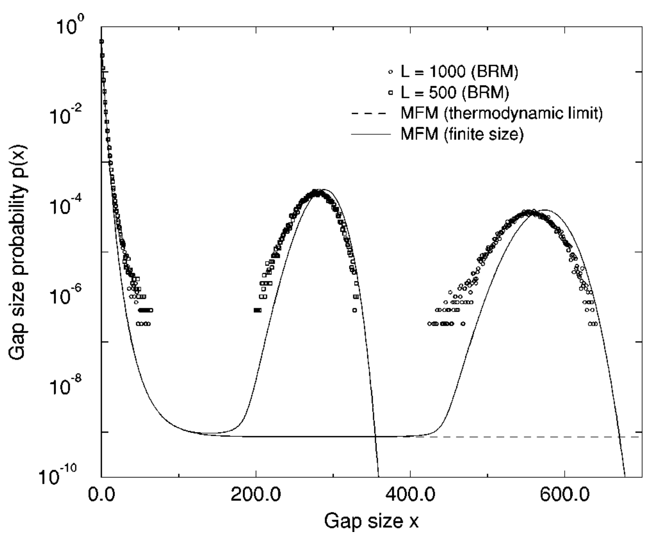

FIG. 9. BRM (simulation) and MFM (calculation) gap size distributions on a linear-log scale for small systems in the jammed regime ($\beta=0.5$, $\lambda=0.02$, $\rho=0.2$). The MFM distribution in the thermodynamic limit is also shown.

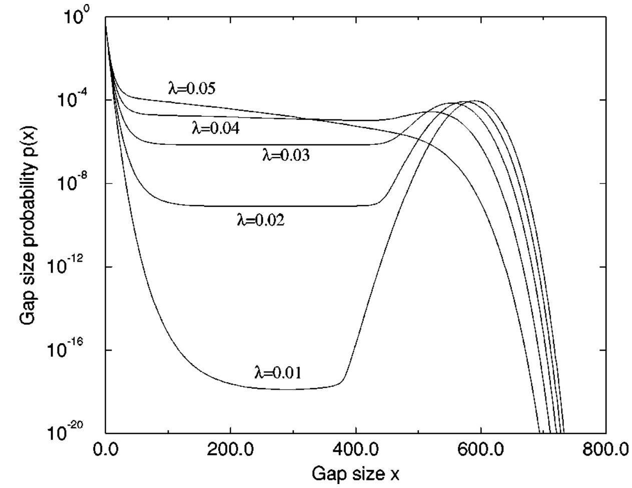

FIG. 10. MFM gap size distributions on a linear-log scale for a small system showing the effect of decreasing λ. The other parameters are $\beta=0.5$, $\rho=0.2$, and $L=1000$.

D. Small λ behavior of MFM

We now examine in detail the behavior of the MFM when λ is small (but nonzero). Figure 10 shows the effect on the MFM gap size distribution of decreasing λ towards zero in a finite system. For $\lambda=0.05$, $p(x)$ decreases monotonically from a maximum at $x=0$. As λ is decreased, $p(x)$ becomes bimodal, signaling the presence of jams. The second peak becomes more pronounced as λ is made smaller; jamming is enhanced.

In order to quantify the behavior of $p(x)$ as $\lambda\rightarrow 0$, we define P_{ext} as the area under the peak in the tail of $p(x)$. Then P_{ext} is the probability of finding an extensive gap in the system and $P_{\mathrm{ext}}\times M$ is the average number of extensive gaps in the system. Figure 11 shows plots of $P_{\mathrm{ext}}\times M$ against M for various values of λ. As λ is decreased from 0.05, one must go to larger systems to observe $P_{\mathrm{ext}}\times M$ falling below 1. This suggests that in the limit $\lambda\rightarrow 0$, a single extensive gap survives in the thermodynamic limit, supporting our claim that a condensation (or jamming) transition does occur in this limit.

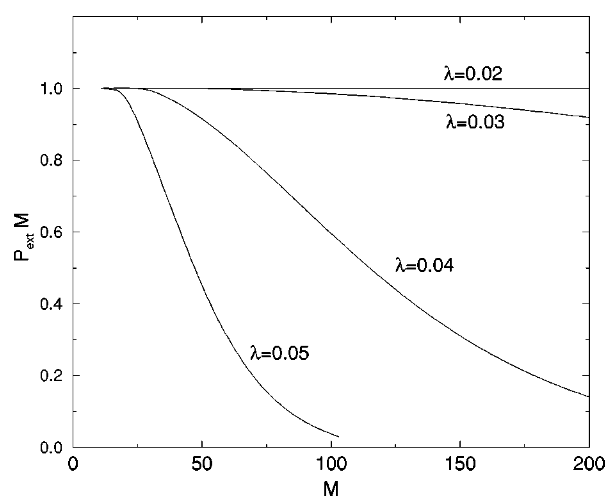

FIG. 11. $P_{\mathrm{ext}}\times M$ against M for the MFM showing the effect of decreasing λ. The other parameters are $\beta=0.5$ and $\rho=0.1$.

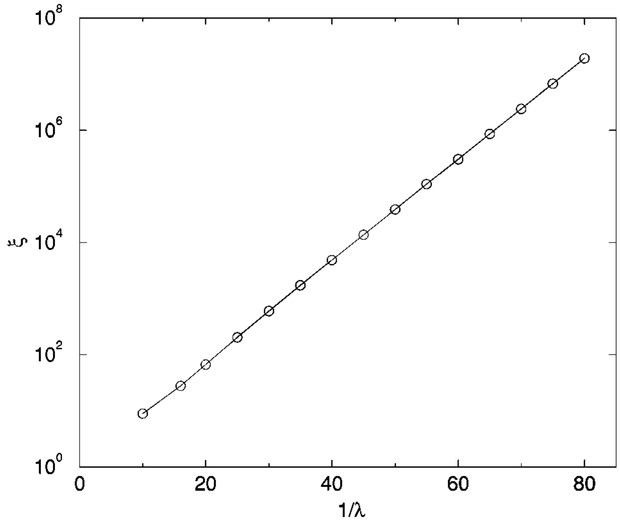

FIG. 12. Linear-log plot of the typical large gap size ξ against $1/\lambda$ for the MFM in the thermodynamic limit. The density is fixed at $\rho=0.3$ and β is 0.5. The lines are shown to guide the eye.

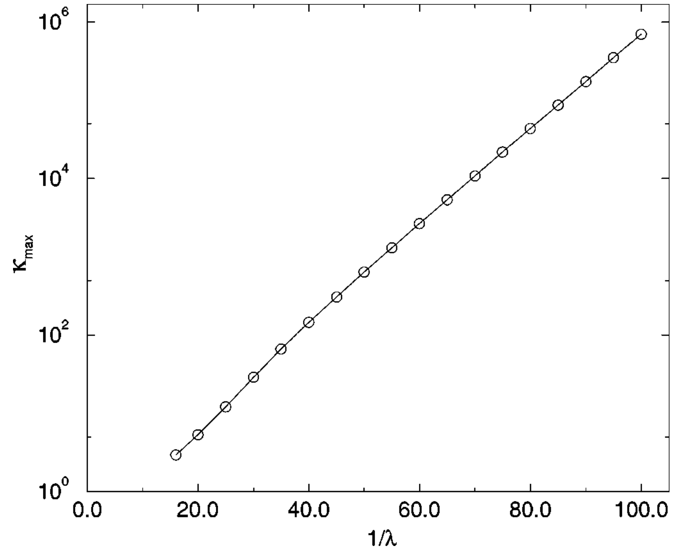

FIG. 14. Linear-log plot of $\kappa_{\max}$ against $1/\lambda$ for the MFM in the thermodynamic limit. β is 0.5. The lines are shown to guide the eye.

We showed in Sec. III A that for the MFM in the thermodynamic limit, $p(x)$ is proportional to $(v/\beta)^x$ for large x. Therefore, the typical size of the large gaps in the system is the decay constant of $p(x)$, given by

$$\xi=\left[\ln\left(\frac{\beta}{v}\right)\right]^{-1}. \tag{28}$$

Figure 12 shows a linear-log plot of ξ against $1/\lambda$ for $\rho=0.3$. We see $\xi\sim\exp(a/\lambda)$ (where a is a constant) for small λ so that the typical size of the large gaps is very large for λ less than about 0.02 and has an essential singularity as $\lambda\to 0$. Since $p(x)$ is sharply peaked at $x=0$ (see Figs. 9 and 10), we deduce that for low density and small λ, a MFM system comprises large clusters of buses that are typically a distance ξ apart. Clearly, this can only apply if $L\gg\xi$; if this is not the case, then the gap size distribution must be bimodal as seen in Fig. 10.

Figure 13 shows the effect on $v(\rho)$ of decreasing λ towards zero in the thermodynamic limit. As λ is decreased, a ''corner'' becomes apparent at a density slightly below 0.5. This behavior is also observed in the BRM (see Fig. 4). While we know that there is no singularity in $v(\rho)$ for $\lambda>0$, one can see that as $\lambda\to 0$, $v(\rho)\to\beta$ for $\rho<0.5$ and $v(\rho)\to 1-\rho$ for $\rho>0.5$. We now examine the sharpness of the crossover from the low density jammed regime to the high density homogeneous regime for small λ.

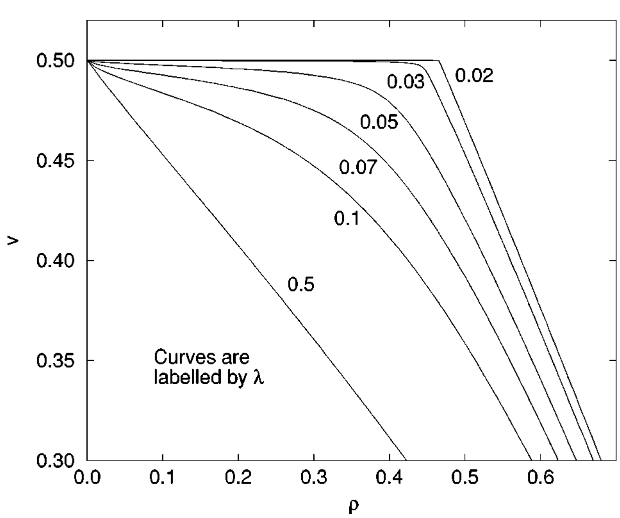

FIG. 13. Velocity as a function of density for the MFM in the thermodynamic limit showing the effect of decreasing λ. β is 0.5.

To estimate the sharpness of the crossover for a given set of parameters, we calculate $\kappa_{\max}$, which we define to be the maximum value of $|v''(\rho)|$. Figure 14 shows a linear-log plot of $\kappa_{\max}$ against $1/\lambda$. For λ less than about 0.02, we see that $\kappa_{\max}$ goes as $\exp(b/\lambda)$, where b is a constant. Therefore, although a strict phase transition occurs only in the limit $\lambda\to 0$, the crossover is exponentially sharp in $1/\lambda$ for small λ; this may be compared with the typical large gap size ξ discussed above, which is also exponentially large in $1/\lambda$. For practical purposes, the crossover may be indistinguishable from a phase transition as is already the case for $\lambda=0.02$ (see Figs. 4 and 13). Note that, as mentioned previously, the thermodynamic and $\lambda\to 0$ limits do not commute: for a finite system the behavior at $\lambda=0$ is trivial (there are no passengers in the system). This means that in a large but not infinite system, the sharp crossover will, as λ is reduced, resemble a phase transition most strongly for some nonzero λ, $\lambda^*(L)$, and then fade away for $\lambda\ll\lambda^*$. (Heuristically, we expect λ^*L to be of order unity.)

IV. COARSENING

In this section we discuss the approach to the steady state in the jamming regime. We have already seen in Fig. 3 that, in finite systems, coarsening of bus clusters (or equivalently the gaps between clusters) occurs for small λ and low bus density. Here, by relating the model to a reaction-diffusion process, we argue that the typical size of the large gaps in the system should eventually grow as $t^{1/2}$ and we provide numerical evidence for this. We also study numerically the relaxation of the average particle velocity $v(t)$ and find that it decays as a power law with an exponent close to -1.

The gap size distribution in the jamming regime can be considered as a superposition of the small gaps inside bus clusters and the large gaps between these clusters. We define $r(t)$ to be the mean size of the large gaps. Recall first from Sec. III that in a mean-field approximation we expect the probability that a particle hops into a gap of size r to be

$$u(r)=\beta+(1-\beta)\exp(-\lambda r/v). \tag{29}$$

If λr is sufficiently large, the motion of a typical bus cluster therefore becomes uncorrelated with the motion of the cluster ahead. When two clusters come sufficiently close together, they coalesce and form a single cluster. The larger $r(t)$ becomes, the more this coalescence is diffusion dominated (i.e., correlations in the motion of clusters are reduced) and the more it resembles a driven $A+A\rightarrow A$ reaction-diffusion process [32] with bus clusters taking the place of A particles. In the driven $A+A\rightarrow A$ process, the characteristic length scale [of which $r(t)$ is an example] grows as $t^{1/2}$. (For the $A+A\rightarrow A$ process, the presence of a preferred direction does not change the scaling from that of the undriven system [31].) We therefore expect $t^{1/2}$ growth in the BRM when $\lambda r(t)$ is sufficiently large, i.e., at sufficiently late times.

We have performed simulations on large systems ($L=5\times10^5$) to investigate the approach to the steady state. In these simulations, inspection of the complete gap size distribution shows that, after a short time, there is a very deep minimum at a gap size of about 20; we observe a negligible number of gaps of this size. Therefore, in the results presented below we have defined a large gap as one with size greater than 20 and $r(t)$ is then the mean size of gaps greater than 20. In discussing the coarsening in a disordered driven diffusive model, Krug and Ferrari [20] use the variance of the complete gap size distribution as their measure of the typical length scale. We have also studied this quantity and we find that its time evolution is entirely consistent with that of $r(t)$, as one would expect in a scaling regime.

Figure 15 shows log-log plots of $r(t)$ obtained from simulation. Close inspection of the $r(t)$ curves reveals that they are straight lines at early times, indicating power law growth with exponent $\simeq 0.3$. By early times we mean the interval $3000<t<10\,000$ for the two larger values of λ and $10\,000<t<30\,000$ for $\lambda=0.005$. However, at later times we see different behavior as the curves become somewhat concave, particularly for the two larger values of λ.

Since we anticipate that $t^{1/2}$ growth will occur only after $r(t)$ has become sufficiently large for bus clusters to be uncorrelated, we expect $r(t)$ to be of the form

$$r(t)=r_0(\lambda)+At^{1/2} \tag{30}$$

at late times. The constant A should depend only very weakly on λ if the coarsening process is truly diffusion dominated. However, r_0 is a measure of how far apart bus clusters must become to be uncorrelated for a particular value of λ, and so we expect it to vary as $\sim 1/\lambda$ from Eq. (29).

Figure 16 shows log-log plots of $r(t)-r_0$ for the same data presented in Fig. 15. The parameter r_0 was estimated by

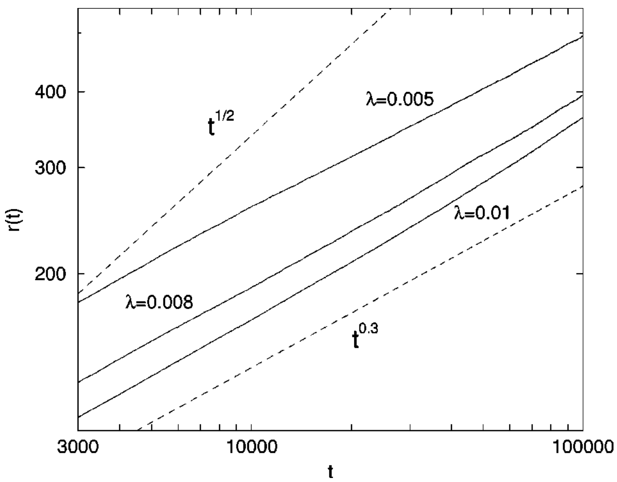

FIG. 15. Log-log plots of the average large gap size $r(t)$ for small values of λ. Simulations had $L=500\,000$, $\rho=0.2$, and $\beta=0.2$. Results for $\lambda=0.01$ and 0.005 were averaged over 15 independent runs and for $\lambda=0.008$, 8 runs. At $t=0$ the buses were positioned at random and there were no passengers. The dashed lines ($t^{1/2}$ and $t^{0.3}$) are shown to guide the eye.

fitting the function (30) to the data for $t>40\,000$. For the two larger values of λ we see that the curves collapse at late times as expected. The behavior at these late times is certainly consistent with $t^{1/2}$ growth. However, the results for $\lambda=0.005$ do not fit this picture. We believe that this is because the $t^{1/2}$ growth regime has not yet been reached for this value of λ. It appears that as λ becomes smaller, the regime characterized by approximate $t^{0.3}$ growth increases in duration. Our results suggest that the BRM may exhibit a crossover from power law growth of $r(t)$ with exponent $\simeq 0.3$ to growth with exponent 1/2. We remark that the MFM exhibits similar, but not identical, coarsening behavior.

We now discuss the behavior of the average velocity $v(t)$ as the steady state is approached. We believe that the true steady-state velocity in the jammed regime of the BRM is very close to, but slightly smaller than, β. Figure 17 shows the decay of $v(t)$ towards its steady-state value. The data, although noisy, show clearly that $v(t)$ relaxes towards β

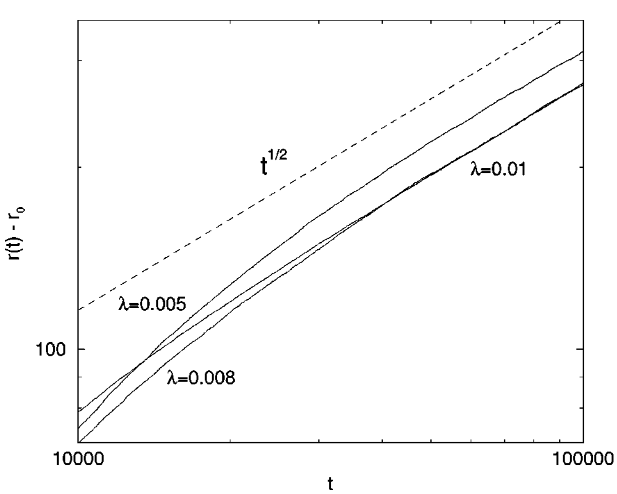

FIG. 16. Log-log plots of $r(t)-r_0$ for small values of λ. The data are the same as that presented in Fig. 15. The dashed line ($t^{1/2}$) is shown to guide the eye.

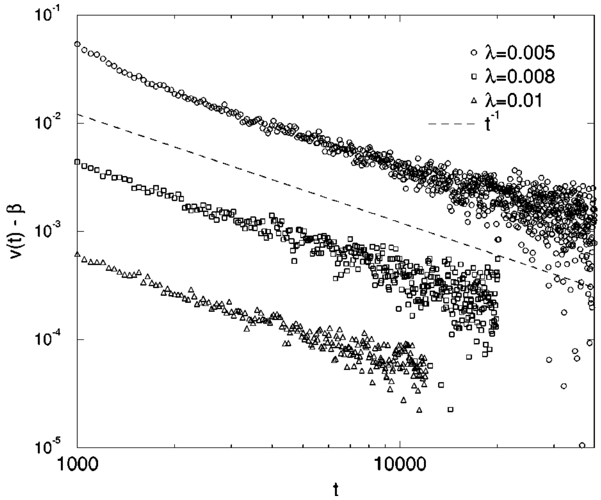

FIG. 17. Log-log plots of $v(t)$-β for small values of λ. Simulations are the same as those for the data shown in Figs. 15 and 16. The dashed line (t^{-1}) is shown to guide the eye. The data have been rescaled vertically for clarity.

$=0.2$ as a power law with an exponent close to -1. The data for later times, although very noisy and not shown, are consistent with t^{-1}. We have no evidence for a crossover in the relaxation of $v(t)$ [in contrast to $r(t)$] but, owing to the quality of our data, neither can we rule out this possibility.

It is interesting to note that we observe coarsening in a system that does not strictly phase separate. We have already argued that for small λ the thermodynamic singularities in the limit $\lambda\to 0$ are replaced by behavior that is smooth but has crossovers that are exponentially sharp in 1/λ. Apparently the $\lambda\to 0$ phase transition likewise shows up in the dynamics where the system appears to phase separate, but only up to a finite (but very long) time so that a truly inhomogeneous state is never reached. Our study of the MFM for small λ in Sec. III D suggests, however, that the final ''homogeneous'' state comprises large clusters of buses separated by gaps having a typical size of order exp(1/λ); hence $t^{1/2}$ growth of the typical large gap size will occur up to some time exponentially large in 1/λ (for a sufficiently large system). Perhaps not surprisingly, this time scale is of the same order of magnitude as our estimate of the time scale on which individual clusters are stable, derived within the two-particle approximation in Sec. II A.

V. DUAL MODEL: CLOGGING

The above completes our study of the BRM. However, as mentioned in Sec. I, our motivation for studying the model is mainly connected with its interesting generic behavior (a driven system with one conserved and one nonconserved variable) rather than its applicability to public transport. In this section we present an alternative interpretation of the model that further reveals its generic behavior.

One can interpret the BRM in a different way by noting that each time a bus hops to the right, the site (hole) that the bus hops into moves to the left. By considering the ''holes'' (henceforth we call them *particles*) to be the moving entities in the model and the ''buses'' to be empty sites, one defines a dual model.

A feature of the dual model is that the nonconserved variable can now be thought of as being ''attached'' to the moving particles rather than fixed sites as in the original interpretation. The nonconserved variable is μ_i, the speed of particle i, which can either be fast ($\mu_i=1$) or slow ($\mu_i=\beta$). If the dynamics of this dual model is to be exactly the same as that of the BRM, then a particle should attempt to hop to the left when the site to its left is updated. However, a more natural dynamics is to only choose particles for update. The full update rules in the latter case are as follows: (1) Pick a particle i at random. (2) If $\mu_i=1$ then $\mu_i\to\beta$ with probability λ. (3) If there is no particle on the site to the left of particle i, then it hops to the left with probability μ_i. (4) If particle i hops, then $\mu_i\to 1$. This model describes a system of particles each of which can exist in two states of mobility. A particle has probability λ of switching to the less mobile state each time step for which it remains stationary. When it finally does move, it is restored to the more mobile state.

While the dual model is not identical to the BRM, we have checked that the numerical behavior is indeed very similar. The density of particles is $1-\rho\equiv\rho'$ and we note that a gap in the BRM is equivalent to a cluster of adjacent particles in the dual model. Thus, the mean-field theories for the two models are equivalent, so long as the hopping rate into a gap of size x in the MFM is interpreted as the hopping rate of a particle leaving the left edge of a cluster of size x in the dual mean-field theory. Jamming becomes a *high density* phenomenon here, characterized by the presence of large clusters of particles. This restores to the word ''jamming'' a meaning closer to that used in everyday life.

We define the average speed of particles as v'. Since in the BRM the magnitude of the bus current must be equal to the magnitude of the hole current, we have

$$v'=\left(\frac{\rho}{1-\rho}\right)v(\rho)=\left(\frac{1-\rho'}{\rho'}\right)v(1-\rho'). \tag{31}$$

The dual model can be thought of as describing stop-start traffic flow with the particles representing cars. The longer a car is at rest (usually because a car in front is blocking it), the more likely it is that the driver will be slow to react when it is possible to move again. Now λ is the parameter determining the strength of the ''slow-to-start'' behavior of cars. This is similar to several ''slow-to-start'' cellular automaton traffic models studied recently [33,34]. In the study of traffic flow, the so-called ''fundamental diagram'' (the current $v'\rho'$ as a function of density ρ') is commonly examined. Figure 18 shows the fundamental diagram for the dual model (simulation and MFM) for different values of λ. One can see that for high densities and λ very small, the current is significantly less than that for $\lambda=0$. It follows from the behavior as $\lambda\to 0$ that in the model, an infinitesimal probability for cars to become slow to start results in a macroscopically large decrease in the current at high density.

A different interpretation of the dual model provides a simple picture of another familiar kind of jamming that might be called ''clogging.'' Consider particles suspended in a fluid that is pumped along a narrow pipe. Imagine that the particles, which are comparable in size to the pipe width (so they cannot pass each other), have some tendency to stick to

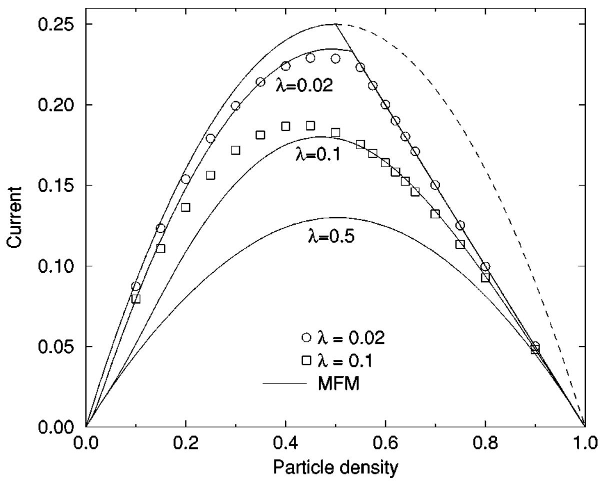

FIG. 18. Particle current as a function of density for dual model for different values of λ. For all simulation data, $L=10\,000$ and MFM results are in the thermodynamic limit. The uppermost solid curve is the MFM result in the thermodynamic limit followed by the limit $\lambda\rightarrow 0$. The dashed curve is the exact result when λ is set equal to zero before the thermodynamic limit is taken. The latter two curves are identical for $\rho'<0.5$.

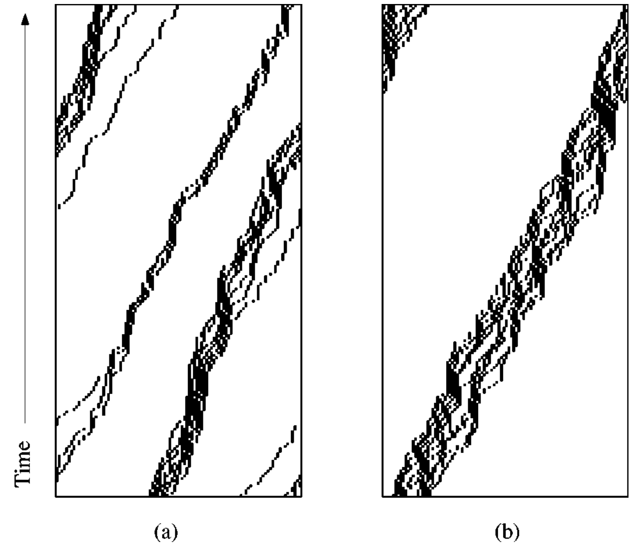

FIG. 19. Space-time plots of bus positions for $\rho=0.2$, $\beta=0.5$, and $L=100$. In (a) $\lambda=0.1$ and in (b) $\lambda=0.02$. There is 1 time step between each snapshot on the time axis. The system is in the steady state.

the walls of the pipe, but that this process takes time to occur and therefore can only happen if a particle remains stationary with respect to the pipe wall for a significant time. A stuck particle will remain stuck if there is another such particle in front of it. If not, the stuck particle will detach from the wall and move on, but only after a delay. This offers an interpretation of the dual model in which the attachment rate of a stationary particle is λ and the rate at which a stuck particle will detach and move on is β. (The parameter α is set by the flow rate of the fluid.) In the limit $\lambda\rightarrow 0$, a phase transition then arises from a homogeneous phase in which the particles move quickly, to a jammed phase of inhomogeneous, slow-moving particles. This transition occurs when the particle density is raised above $1-\rho_c$, where ρ_c was defined earlier for the BRM.

More generally, such a model could be taken as a highly simplified discrete description of any fluid that has a tendency to clog. There are many fluids that will solidify when at rest but remain in a fluid state if kept moving rapidly enough—examples include colloid/polymer mixtures [35], clay gels (used as drilling muds in the oil industry) and, in some circumstances, blood. A similar description might apply to particulate suspensions (say in a horizontal pipe), which, if not kept moving, will settle under gravity into a relatively immobile deposit.

VI. RELEVANCE TO REAL BUSES

While the BRM is not designed to model a real bus route with any great accuracy, it is worth commenting on its possible relevance to the bus route problem. To do this, we must relate our model parameters to those of a real (in our case, Scottish) bus route. L is the number of stops on the circular route and the bus density ρ is the number of buses per stop. We expect ρ to be quite low (perhaps 0.1), and hence to be in the regime where jamming could potentially occur. The parameter α, which we take to be 1, is inversely proportional to the average time a bus takes to travel from one stop to the next (with no stopping for passengers). The parameters β and λ are defined relative to α. The amount by which buses are delayed owing to their having to pick up passengers is reflected in β. Our choice of $\beta=0.5$ is, we believe, a reasonable figure for a city bus route—a bus picking up passengers at every stop will progress about half as quickly as one that has no passengers to collect. We interpret λ as the probability that one or more passengers arrive at a typical bus stop in the time it takes a bus to go from one stop to another (roughly a minute, say). Of course, there is no such thing as a typical bus stop on a city bus route but we believe that λ should often lie between 0.1 and 1, depending on location, time of day, and other factors.

In our examination of the BRM up to this point, we have primarily studied the limit of large system size L. For real bus routes, however, we expect L, the number of bus stops, to be in the region of 50 to 100. Figure 19(a) shows the positions of 10 buses in a system of 100 bus stops for 200 time steps in the steady state, with $\lambda=0.1$. Transient clusters of buses are clearly visible in the system, a scenario familiar to regular users of the bus route. A similar space-time plot is shown for $\lambda=0.02$ in Fig. 19(b). As was the case for larger systems, we see a single large cluster of buses that is relatively stable. Our model suggests, therefore, that such catastrophic clusters of buses will not form if the arrival rate of passengers is sufficiently large, but that at some intermediate arrival rates, a given bus is quite likely to be part of a small cluster. Of course, as the arrival rate is increased still further, the BRM predicts an increasingly disordered bus route with little clustering.

Finally, we wish to comment on the implications of a ''hail-and-ride'' bus system, where there are no fixed bus stops and passengers may hail a bus at any point on the route. In the BRM, this corresponds to a system with a large number of stops (large L) and very small values of both λ and β. Hence, one expects the buses to cluster very strongly, much as in Fig. 7 (with a much lower bus density, however).

Compounding this problem, with β very small the leading bus in a cluster moves extremely slowly. Thus, the use of a ''hail-and-ride'' system, while on the surface more convenient for passengers, would have dire consequences for the efficiency of the bus service. However, it is worth pointing out that on routes used by very few passengers, λ might be so small that one would observe essentially $\lambda=0$ behavior, i.e., a homogeneous system with buses being scarcely delayed. This is because the probability that a site has passengers on it cannot be greater than λL; if $\lambda L \ll 1$, no bus is significantly delayed by having to pick up passengers.

VII. DISCUSSION

In this work we have provided strong evidence that the BRM undergoes a jamming transition as a function of the density ρ of buses in the limit where λ, the rate of arrival of passengers, tends to zero. This provides an example of a homogeneous system with local, stochastic dynamics that exhibits a jamming transition in one dimension. Although it remains an open problem to find an exact solution of the model, we have shown that the steady state of a mean-field approximation is solvable. This approximation captures the essence of how the ordering into jams occurs—the density of passengers at a site provides information about the time at which a bus last visited the bus stop. From the elapsed time it is inferred how far away the next bus is along the route. Thus the passengers mediate an effective long range interaction between the buses. The nature of the mean-field approximation is to replace this ''induced'' interaction (which is subject to stochastic variation) with a deterministic one.

The transition also has strong analogies with the jamming transition induced by a single defect particle or disorder [13–20]. In that case it has been shown that the transition is reminiscent of Bose condensation [19]. In the present model an interesting point is that there is no defect bus present; all buses are equal. Instead, the system spontaneously selects a bus behind which a jam forms and the waiting passengers condense in front of this bus. As mentioned in Sec. I, it would make very little difference if overtaking of buses were allowed in the model, since its effect would be merely to interchange the leading two buses of a jam. The new leading bus would then proceed as slowly as its predecessor. This makes the physics different from the defect mediated case where unhindered overtaking would prevent a jam from ever arising. We have shown, at least in the two body approximation of Sec. II A, that the time for the lead bus to escape from its jam diverges strongly with system size. This may be compared with the ''flip time'' in another model exhibiting spontaneous symmetry breaking [7].

The bus route model has an interesting dual model, described in Sec. V. This considers the spaces between buses to be moving entities. If the moving entities are interpreted as cars, the dual model is a particular type of slow-to-start traffic model [33,34]. If the moving entities are particles suspended in a fluid, the dual model is a model of ''clogging'' in the transport of sticky particles, or a gelling fluid, down a pipe. In either interpretation, there is (in the limit of small λ) a phase transition between a homogeneous and an inhomogeneous phase; however, the inhomogeneous (jammed) phase now arises at high density. This contrasts to the BRM

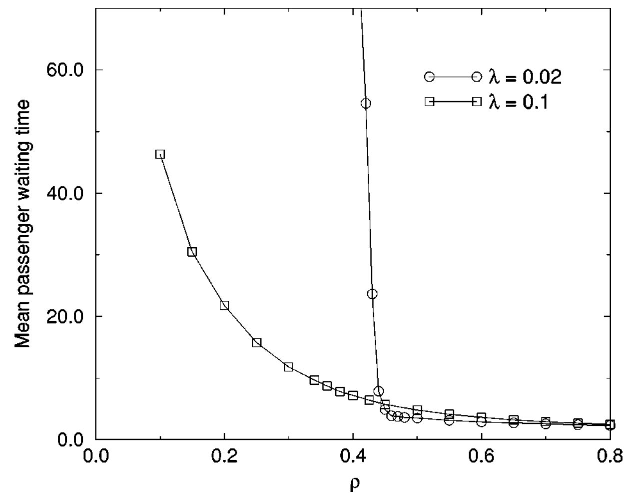

FIG. 20. Plot of the mean passenger waiting time as a function of bus density for two different values of λ. The simulation parameters are $L=10\,000$ and $\beta=0.5$.

itself and related models [13–20] of defect-mediated jamming, for which jams arise when the vehicle density is too low.

The dual model is, in our view, interesting because it describes the jamming of particles whose mobility depends on an internal dynamical degree of freedom. This is represented by the nonconserved variable μ. In the dual model itself, the nonconserved variable keeps an internal record of how long it is since the particle last moved. More generally however, other models of this type could entail a nonconserved internal variable representing, say, the orientation of rodlike particles. (Such particles would be much more likely to jam in some orientations than others.) We have shown that the existence of an additional, nonconserved degree of freedom is, at least in a specified limit ($\lambda \to 0$), enough to cause a symmetry-breaking phase transition to a jammed state, in a homogeneous one-dimensional driven system.

We now comment on our results for nonzero λ. Although we argue that a transition only occurs as $\lambda \to 0$, we have shown that a strong vestige of the transition remains for small values of λ. In fact, in the mean-field theory of the BRM, the crossover between the two regimes becomes exponentially sharp in $1/\lambda$. Therefore, in practice the crossover becomes very difficult to distinguish from a strict thermodynamic transition when $1/\lambda$ is of order 100 or more. The $\lambda \to 0$ transition also strongly influences the *dynamical* behavior of the system when λ is small but nonzero: we observed coarsening behavior (presumably transient) as would usually be associated with systems that do strictly phase separate. All this highlights the fact that care is required to unambiguously identify phase transitions, as opposed to crossover phenomena. Indeed, it is possible that closely related crossover phenomena occur in cellular automata models of traffic [36,37].

Finally, we show in Fig. 20 the average time that a passenger is required to wait for a bus. The system size is 10 000, unphysical in terms of real bus routes but representative of the thermodynamic limit. For $\lambda=0.1$, we see that, as one would expect, the average waiting time increases smoothly as the density of buses is reduced. However, for $\lambda=0.02$, it increases very sharply at an intermediate value of the density, and for low densities, the bus service becomes highly inefficient. This figure serves as yet another cruel reminder of the vagaries of the bus route to those whose lives are unfettered by the ownership of a motor car.

ACKNOWLEDGMENTS

O.J.O. is supported by the University of Edinburgh. M.R.E. thanks the Royal Society for financial support.

APPENDIX: STEADY STATE SOLUTION OF MEAN-FIELD MODEL

The MFM comprises M particles hopping on a 1D ring of L lattice sites. The probability of hopping into a gap of size x is $u(x)$ which must obey $u(0)=0$ (hard-core exclusion). The update is random sequential so that the probability of a given particle being updated at each step is $1/M$.

A configuration $\mathcal{C}$ is completely specified by a sequence of gap sizes $\{x_i\}$ where x_i is the size of the gap in front of the ith particle.

In the steady state, the master equation for the system is

$$\sum_{\mathcal{C}'\neq\mathcal{C}} W(\mathcal{C}\to\mathcal{C}')P(\mathcal{C})=\sum_{\mathcal{C}'\neq\mathcal{C}} W(\mathcal{C}'\to\mathcal{C})P(\mathcal{C}'), \quad \text{(A1)}$$

where $P(\mathcal{C})$ is the steady state probability of configuration $\mathcal{C}$ and $W(\mathcal{C}\to\mathcal{C}')$ is the probability of an elementary transition from $\mathcal{C}$ to $\mathcal{C}'$.

First, consider the right-hand side (RHS) of Eq. (A1). Define $\mathcal{C}=\{x_1,\ldots,x_i,\ldots,x_M\}$ and $\mathcal{C}_j=\{x_1,\ldots,x_{j-1}-1,x_j+1,\ldots,x_M\}$. Then the only configurations from which $\mathcal{C}$ can be obtained by an elementary transition are the $\mathcal{C}_j$, with the constraint that $x_{j-1}>0$. We therefore have

$$W(\mathcal{C}_j\to\mathcal{C})=\frac{1}{M}u(x_j+1) \quad \text{(A2)}$$

and the RHS of Eq. (A1) becomes

$$\frac{1}{M}\sum_j u(x_j+1)P(\mathcal{C}_j)\,\theta(x_{j-1}), \quad \text{(A3)}$$

where $\theta(x)$ is the usual Heaviside (step) function.

Now consider the LHS of Eq. (A1). Define $\mathcal{C}_k=\{x_1,\ldots,x_{k-1}+1,x_k-1,\ldots,x_M\}$. The $\mathcal{C}_k$ are the only configurations that can be obtained from $\mathcal{C}$ by an elementary transition, with the constraint that $x_k>0$. Therefore we have

$$W(\mathcal{C}\to\mathcal{C}_k)=\frac{1}{M}u(x_k) \quad \text{(A4)}$$

and the LHS of Eq. (A1) becomes

$$\frac{1}{M}\sum_k u(x_k)P(\mathcal{C})\,\theta(x_k). \quad \text{(A5)}$$

Equating Eqs. (A5) and (A3) gives

$$\sum_k u(x_k)P(\mathcal{C})\,\theta(x_k)=\sum_j u(x_j+1)P(\mathcal{C}_j)\,\theta(x_{j-1}). \quad \text{(A6)}$$

Since periodic boundary conditions apply, the indices on the LHS and RHS of Eq. (A6) can be matched up to give

$$\sum_j u(x_{j-1})P(\mathcal{C})\,\theta(x_{j-1})=\sum_j u(x_j+1)P(\mathcal{C}_j)\,\theta(x_{j-1}). \quad \text{(A7)}$$

To solve this equation, we assume a product form for P. We write

$$P(\{x_i\})=\frac{1}{Z(L,M)}q(x_1)\cdots q(x_i)\cdots q(x_M), \quad \text{(A8)}$$

where $Z(L,M)$ is a normalization. Substituting this product form into Eq. (A7) gives

$$\sum_j\left(\prod_i q(x_i)\right)\theta(x_{j-1})A_j=0, \quad \text{(A9)}$$

where

$$A_j=u(x_{j-1})-\frac{u(x_j+1)q(x_{j-1}-1)q(x_j+1)}{q(x_{j-1})q(x_j)}. \quad \text{(A10)}$$

To solve this for $q(x_j)$, make the ansatz that

$$u(x_{j-1})\frac{q(x_{j-1})}{q(x_{j-1}-1)}=u(x_j+1)\frac{q(x_j+1)}{q(x_j)} \quad \text{(A11)}$$

for $x_{j-1}>0$. This has the solution

$$q(x)=\prod_{y=1}^{x}\frac{1}{u(y)}, \quad \text{(A12)}$$

which, together with Eq. (A8), gives $P(\{x_j\})$ as required.

[1] B. Schmittmann and R. K. P. Zia, *Statistical Mechanics of Driven Diffusive Systems*, Vol. 17, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1995).
[2] K. Nagel, Phys. Rev. E **53**, 4655 (1996).
[3] T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 215 (1995).
[4] R. Lahiri and S. Ramaswamy, Phys. Rev. Lett. **79**, 1150 (1997).
[5] P. Gacs, J. Comput. Sys. Sci. **32**, 15 (1986).
[6] M. R. Evans, D. P. Foster, C. Godrèche, and D. Mukamel, Phys. Rev. Lett. **74**, 208 (1995).
[7] C. Godrèche, J.-M. Luck, M. R. Evans, D. Mukamel, S. Sandow, and E. R. Speer, J. Phys. A **28**, 6039 (1995).
[8] P. F. Arndt, T. Heinzel, and V. Rittenberg, J. Phys. A **31**, L45 (1998).
[9] M. R. Evans, Y. Kafri, H. M. Koduvely, and D. Mukamel, Phys. Rev. Lett. **80**, 425 (1998).

[10] J. Krug, Phys. Rev. Lett. **67**, 1882 (1991).
[11] U. Alon, M. R. Evans, H. Hinrichsen, and D. Mukamel, Phys. Rev. Lett. **76**, 2746 (1996); Phys. Rev. E **57**, 4997 (1998).
[12] J. Kertèsz and D. E. Wolf, Phys. Rev. Lett. **62**, 2571 (1989).
[13] S. A. Janowsky and J. L. Lebowitz, Phys. Rev. A **45**, 618 (1992).
[14] G. Schütz, J. Stat. Phys. **71**, 471 (1993).
[15] B. Derrida, S. A. Janowsky, J. L. Lebowitz, and E. R. Speer, Europhys. Lett. **22**, 651 (1993); J. Stat. Phys. **73**, 813 (1993).
[16] K. Mallick, J. Phys. A **29**, 5375 (1996).
[17] B. Derrida, in *Statphys19*, edited by B.-L. Hao (World Scientific, Singapore, 1996).
[18] Z. Toroczkai and R. K. P. Zia, J. Stat. Phys. **87**, 545 (1997).
[19] M. R. Evans, Europhys. Lett. **36**, 13 (1996); J. Phys. A **30**, 5669 (1997).
[20] J. Krug and P. A. Ferrari, J. Phys. A **29**, L213 (1996).
[21] E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **56**, 6680 (1997).
[22] A. J. Bray, Adv. Phys. **43**, 357 (1994).
[23] E. Ben-Naim, P. L. Krapivksy, and S. Redner, Phys. Rev. E **50**, 822 (1994).
[24] T. Nagatani, Phys. Rev. E **51**, 922 (1995).
[25] F. Ritort, Phys. Rev. Lett. **75**, 1190 (1995).
[26] P. Bialas, Z. Burda, and D. Johnston, Nucl. Phys. B **493**, 505 (1997).
[27] J.-M. Drouffe, C. Godrèche, and F. Camia, J. Phys. A **31**, L19 (1998).
[28] H. Kramers, Physica (Amsterdam) **7**, 284 (1940).
[29] F. Spitzer, Adv. Math. **5**, 246 (1970).
[30] E. D. Andjel, Ann. Prob. **10**, 525 (1982).
[31] V. Privman, A. M. R. Cadilhe, and M. L. Glasser, J. Stat. Phys. **81**, 881 (1995).
[32] For a review see D. ben-Avraham, in *Nonequilibrium Statistical Mechanics in One Dimension*, edited by V. Privman (Cambridge University Press, Cambridge, 1997).
[33] S. C. Benjamin, N. F. Johnson, and P. M. Hui, J. Phys. A **29**, 3119 (1996).
[34] A. Schadschneider and M. Schreckenberg, Ann. Phys. (Leipzig) **6**, 541 (1997).
[35] W. C. K. Poon, A. D. Pirie, M. D. Haw, and P. N. Pusey, Physica A **235**, 110 (1997).
[36] B. Eisenblätter, L. Santen, A. Schadschneider, and M. Schreckenberg, Phys. Rev. E **57**, 1309 (1998).
[37] M. Sasvári and J. Kertész, Phys. Rev. E **56**, 4104 (1997).
[38] O. J. O'Loan, M. R. Evans, and M. E. Cates, Europhys. Lett. **42**, 137 (1998).